Chapter 2

Integral Calculus

2.1 Differentiation Under Integration

2.1.1 Leibniz Integral Rule

- $\frac{d}{dx}\Big[\int_a^b f(x,t)dt\Big] = \int_a^b \frac{\partial}{\partial x} f(x,t)dt.$
- $\frac{d}{dx}\Big[\int_{a(x)}^{b(x)} f(x,t)dt\Big] = f(x,b(x))\frac{d}{dx}b(x) - f(x,a(x))\frac{d}{dx}a(x) + \int_{a(x)}^{b(x)} \frac{\partial}{\partial x} f(x,t)dt.$
- $\frac{d}{dx}\Big[\int_{a(x)}^{b(x)} f(t)dt\Big] = f(x,b(x))\frac{d}{dx}b(x) - f(x,a(x))\frac{d}{dx}a(x).$

2.1.2 A Working Procedure

▶ Suppose $g(x) = \int_x^{x^2} f(t)dt$ then to find $g'(x)$, we take $\int f(t)dt = \phi(t) \Rightarrow \phi'(t) = f(t)$. Now $g(x) = \phi(x^2) - \phi(x) \Rightarrow g'(x) = 2x\phi'(x^2) - \phi'(x) = 2xf(x^2) - f(x)$.

[Do It Yourself] 2.1. *If* $F(x) = \int_{x^3}^{4} \sqrt{4+t^2}dt,\ x \in \mathbb{R}$. *Then* $F'(1)$ *equals*
(A) $-3\sqrt{5}$. *(B)* $-2\sqrt{5}$. *(C)* $2\sqrt{5}$. *(D)* $3\sqrt{5}$.

[Do It Yourself] 2.2. *Let* $f : [0,1] \to \mathbb{R}$ *be a function defined as* $f(t) = t^3\big[1 + \frac{1}{5}\cos(\ln t^4)\big]$, *for* $t \in (0,1]$. *Let* $F : [0,1] \to \mathbb{R}$ *be defined as* $F(x) = \int_0^x f(t)dt,\ x \in \mathbb{R}$ *Then* $F''(0)$ *equals*
(A) 0. *(B)* $3/5$. *(C)* $-5/3$. *(D)* $1/5$.

[Do It Yourself] 2.3. *If* $\int_0^x f(t)dt = x^2 \sin x + x^3$. *Then find* $f(\pi/2)$.

[Do It Yourself] 2.4. *Let* $f : [0,1] \to \mathbb{R}$ *be defined by*

$$f(t) = \begin{cases} 0 & \text{if } 0 < t \leq 1 \\ e^{x^2} - e & \text{if } t = 0 \end{cases}$$

Now, define $F : [0,1] \to \mathbb{R}$. *by* $F(x) = \int_0^x f(t)dt$. *Then* $F''(0)$ *equals to*
(A) 0. (B) $\frac{3}{5}$. *(C)* -$\frac{5}{3}$. *(D)* $\frac{1}{5}$.

[Do It Yourself] 2.5. *Let* $F(x) = \int_0^x e^t(t^2 - 3t - 5)dt$, $x > 0$. *Then find the number of roots of* $F(x) = 0$ *in the interval* $(0,4)$. $(Ans : 0)$

[Do It Yourself] 2.6. *The value of the limit* $\lim_{x \to \frac{1}{2}} \dfrac{\int_{\frac{1}{2}}^x \cos^2(\pi t)dt}{\frac{e^{2x}}{2} - e(x^2 + \frac{1}{4})}$ *is*
(A) 0. *(B)* π/e. *(C)* $\pi^2/2e$. *(D)* $-\pi^2/2e$.

2.2 Application of Integration

▶ We will study the computation of area, length of a curve, surface integral, volume integral and surface of revolution.
▶ Tangent of the curve $y = f(x)$ at (α, β): $\boxed{(y - \beta) = \frac{dy}{dx}|_{(\alpha,\beta)}(x - \alpha)}$.
▶ Asymptote: We will understand this concept using graphs. Sometimes asymptotes are helpful to draw graphs e.g. $y = 1/x$, $y = \frac{1}{x} + 1$.

▶ The curve $f(x, y) = 0$ is known as implicit form. The curve $y = f(x)$ or, $x = g(y)$ is known as explicit form. The curve $x = f_1(t)$, $y = f_2(t)$ is known as parametric form.

▶ Suppose we want to draw the curve $f(x, y) = 0$.

▶ Rule 1: If y^2 is present $\Rightarrow$ curve is symmetric about $x - axis$.

▶ Rule 2: Try to find its asymptotes.

▶ Rule 3: If the curve is unchanged under $x, y \Rightarrow$ curve is symmetric about the line $y = x$.

▶ Rule 4: Try to find some point on $x - axis$, $y - axis$ by putting $y = 0$, $x = 0$. Also check the position of origin $(0, 0)$.

▶ Rule 5: For explicit curve you can check about its monotonicity.

2.2.2 Curve Plotting

- A $\boxed{\text{Circle}}$: $x^2 + y^2 = a^2,\ x^2 + y^2 = 2^2,\ (x-2)^2 + y^2 = 3^2,\ (x-1)^2 + (y-2)^2 = 3^2.$
- A $\boxed{\text{Parabola}}$: $y^2 = \pm 4ax,\ x^2 = \pm 4.2.y,\ (y-1)^2 = \pm 4.3.(x-2).$
- A $\boxed{\text{Ellipse}}$: $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1,\ \frac{x^2}{3^2} + \frac{y^2}{2^2} = 1,\ \frac{x^2}{2^2} + \frac{y^2}{3^2} = 1.$
- A $\boxed{\text{Hyperbola}}$: $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1,\ \frac{x^2}{3^2} - \frac{y^2}{2^2} = 1,\ \frac{x^2}{2^2} - \frac{y^2}{3^2} = 1,\ x^2 - y^2 = 1,\ xy = 2^2.$

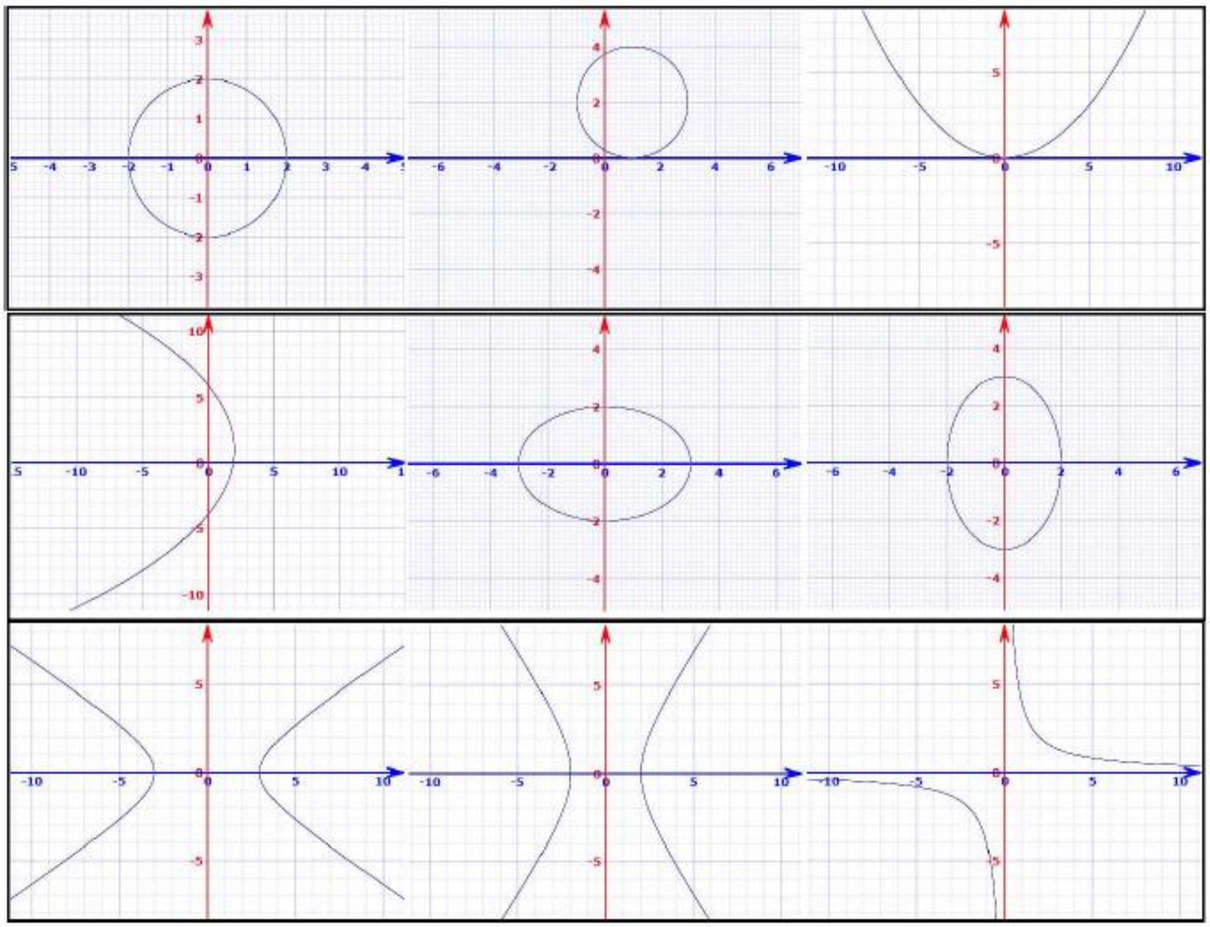

Figure 2.1: Identify The Curves: Circle, Parabola, Ellipse, Hyperbola.

▶ B Catenary: $y = a\cosh(\frac{x}{a})$. Parametric form: $x = c\ln(\sec t + \tan t),\ y = c\sec t$.

▶ B Folium of Descartes: $x^3 + y^3 = 3axy$. Parametric form: $x = \frac{3at}{1+t^3},\ y = \frac{3at^2}{1+t^3}$.

▶ B Astroid: $x^{2/3} + y^{2/3} = a^{2/3}$. Parametric form: $x = a\cos^3 t,\ y = a\sin^3 t$.

▶ B Cissoid: $y^2(a - x) = x^3$. Parametric form: $x = \frac{at^2}{1+t^2},\ y = \frac{at^3}{1+t^2}$.

▶ B Strophoid: $(x^2 + y^2)x = a(x^2 - y^2),\ a > 0$. Parametric form: $x = \frac{a(1-t^2)}{1+t^2},\ y = \frac{at(1-t^2)}{1+t^2}$.

▶ B Semi-cubical Parabola: $ay^2 = x^3,\ (a > 0)$. [From Catenary to Inverted Cycloid: $a = 2$]

▶ B Witch of Agnesi: $xy^2 = 4a^2(2a - x)$.

▶ B Cycloid: Parametric form: $x = a(t - \sin t),\ y = a(1 - \cos t)$.

▶ B Inverted Cycloid: Parametric form: $x = a(t + \sin t),\ y = a(1 - \cos t)$.

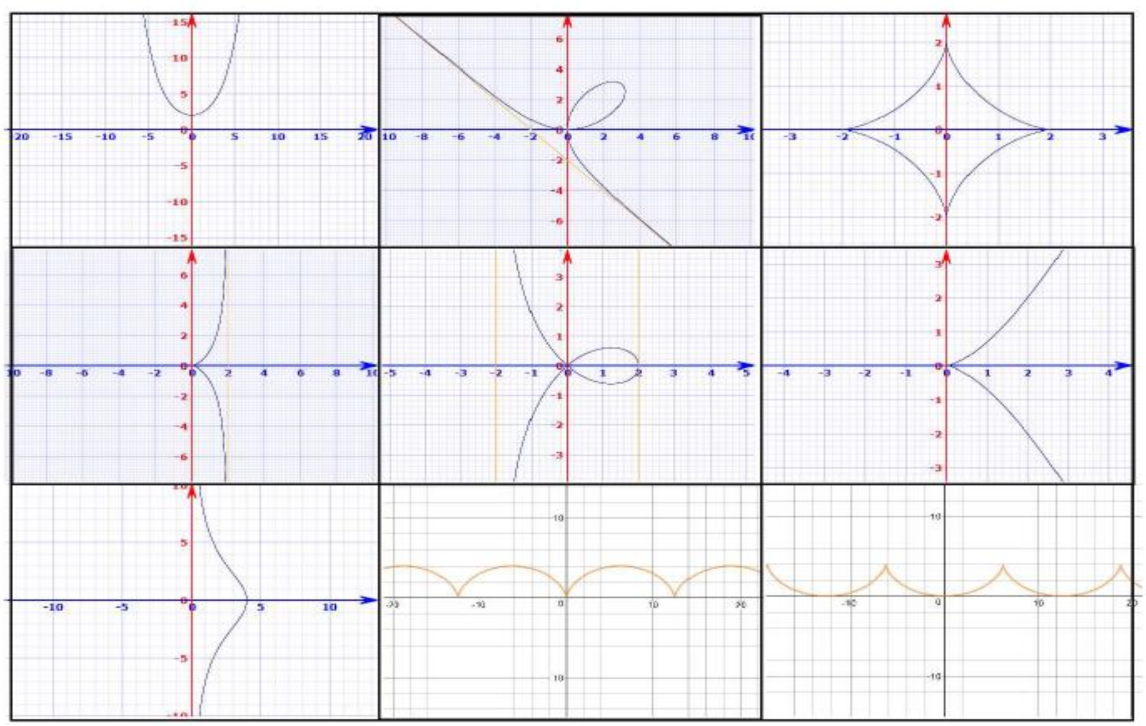

Figure 2.2: The Curves are: Catenary, Folium of Descartes, Astroid, Cissoid, Strophoid, Semi-cubical Parabola, Witch of Agnesi, Cycloid, Inverted Cycloid. Also see the asymptotes.

▶ C A two parameter curve: $a^3y^2 = x^4(b+x),\ a=2,\ b=3.$

▶ C A one parameter curve: $x^2y^2 = a^2(y^2 - x^2),\ a=2.$

▶ Now we will study some polar curve. For each curve we assume $a = 2$.

▶ C Cardiode: $r = a(1-\cos\theta)$ [In figure], $r = a(1+\cos\theta)$.

▶ C Circle: $r = 2a\sin\theta$.

▶ C Lemniscate of Bernoulli: $r^2 = a^2\cos 2\theta$.

▶ C Rose Petals: $r = a\sin n\theta$. If $n = odd \Rightarrow No.\ of\ leaves\ = n$ and if $n = even \Rightarrow No.\ of\ leaves\ = 2n$

▶ C Some Petals: $r = a\sin 2\theta,\ r = a\sin 3\theta,\ r = a\cos 2\theta,\ r = a\cos 3\theta.$

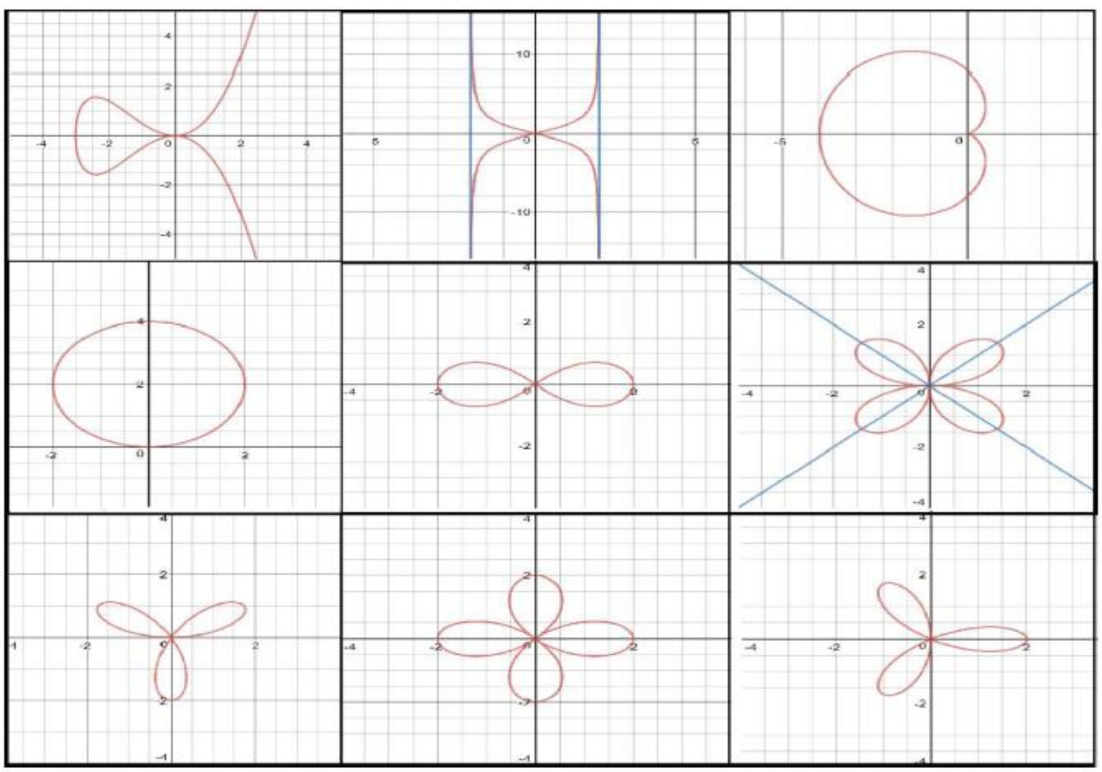

Figure 2.3: The Curves are: Two parameter, One Parameter, Cardiode, Circle, Lemniscate, Petals. In Image 6, the length of leaves within the st. line is a.

■ Although petals are polar curves but here I will give an overview w.r.t. x and $y - axis$.
■ Here $r = a\sin 2\theta$ gets 4 leaves so each co-ordinate gets one leaves but not on axes. For $r = a\sin 4\theta$ gets 8 leaves so each co-ordinate gets two leaves and so on.
■ Here $r = a\sin 3\theta$ gets 3 leaves with one leaves on negative $y-axis$ and other 2 distributed over whole region. For $r = a\sin 5\theta$ gets 5 leaves with one leaves on positive $y - axis$ and other 4 distributed over whole region and so on.
■ Here $r = a\cos 2\theta$ gets 4 leaves so each co-ordinate gets one leaves and each on axes. For $r = a\sin 4\theta$ gets 8 leaves so each 4 axes gets 4 leaves and rest 4 are distributed on gaps and so on.
■ Here $r = a\cos 3\theta$ gets 3 leaves with one leaves on positive $x-axis$ and other 2 distributed over whole region. For $r = a\sin 5\theta$ gets 5 leaves with one leaves on positive $x - axis$ and other 4 distributed over whole region and so on.